Classical and quantum mechanics of the nonrelativistic Snyder model

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Abstract

The Snyder model is an example of noncommutative spacetime admitting a fundamental length scale β and invariant under Lorentz transformations, that can be interpreted as a realization of the doubly special relativity axioms. Here, we consider its nonrelativistic counterpart, i.e. the Snyder model restricted to three-dimensional Euclidean space. We discuss the classical and the quantum mechanics of a free particle in this framework, and show that they strongly depend on the sign of a coupling constant λ, appearing in the fundamental commutators and proportional to β^2. For example, if λ is negative, momenta are bounded. On the contrary, for positive λ, positions and areas are quantized. We also give the exact solution of the harmonic oscillator equations both in the classical and the quantum case, and show that its frequency is energy dependent.

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1. INTRODUCTION

Several years ago, in the attempt to introduce a short distance cutoff in field theory, Snyder proposed a model of noncommutative spacetime, admitting a fundamental length scale and invariant under the Lorentz group [1]. This proposal was ahead of its time and went almost unnoticed, until a few years ago, when noncommutative spacetimes became fashionable, mainly in connection with string theories, where they emerge in certain low-energy limits [2]. Also considerations on quantum gravity and black hole physics seem to indicate that the structure of spacetime must be noncommutative at scales close to the Planck length [3]. In particular, the extremely high energies necessary to resolve very small distances could perturb the spacetime structure by their quantum gravitational effects.

Similar arguments on the structure of spacetime at small length scales were also the basis of the proposal of doubly special relativity (DSR) [4]. This is a model of spacetime admitting a fundamental scale that sets a bound on the allowed values of the momentum, and implies a deformation of the Poincaré symmetry and of the dispersion relations of elementary particles. The natural realization of DSR is on a phase space equipped with a noncanonical symplectic structure [5]. It turns out that the Snyder model can be interpreted as an instance of DSR. In [6], it was shown in fact that the Snyder algebra is a particular realization of the general DSR algebra. In [7], the dynamics of the Snyder model was investigated in the context of DSR, showing that it implies the existence of a maximal allowed value for the mass of a free particle.

The model of spacetime proposed by Snyder is based on the commutation relations[1]

$$[x_\mu, x_\nu] = i\,\lambda\, J_{\mu\nu}, \qquad [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = i\,(\eta_{\mu\nu} + \lambda\, p_\mu p_\nu)\,, \tag{1.1}$$

where λ is a coupling constant, usually assumed to be of the scale of the square of the Planck length, and the $J_{\mu\nu}$ are the generators of the Lorentz algebra. In contrast with the most common models [3], the commutators are not constant, but are functions of the phase space variables, and this allows them to be compatible with the Lorentz symmetry. The algebra (1.1) can be obtained by constraining the momenta to lie on a hypersphere in a (4+1)-dimensional space [1]. From this point of view the Snyder model can be viewed as the equivalent of de Sitter spacetime for momentum space.

In spite of its recent revival, the physical content of the Snyder model has not been investigated in detail. Most investigations have been in fact directed to its formal properties. In [8] its dynamics was derived from a constrained Hamiltonian system. Using similar methods, the authors of [9] obtained the Snyder commutation relations from a six-dimensional setting. In [10] the same techniques were used to study the symmetries of the model.

The most interesting physical implications of the Snyder model are a generalization of the uncertainty relations [6,11], similar to that proposed in [12], implying a lower bound for the uncertainty in position, and the discreteness of the spectra of area and volume [13]. The interpretation of the model in operational terms has also been addressed in [14].

[1] We use the following conventions: Greek indices run from 0 to 3, Latin indices from 1 to 3, the metric signature is $(-,+,+,+)$.

Till now, all the investigations have been restricted to the case $\lambda > 0$. However, the physics strongly depends on the sign of the coupling constant. In particular, the case $\lambda < 0$, that we call anti-Snyder in analogy with anti-de Sitter, is the one relevant for DSR, since it implies an upper bound on the mass of free particles [7]. It is therefore interesting to compare the two possibilities.

In this paper we shall consider the properties of a 3-dimensional Euclidean version of the Snyder model. This can be interpreted as a restriction of the original model to its spatial sections, or more properly as a nonrelativistic version of the model. The interest of limiting our considerations to the nonrelativistic model relies in the fact that, while the main features (noncommutativity of the geometry, generalized uncertainty relations) of the relativistic model are maintained, one can easily implement quantum mechanics, whereas the definition of a relativistic quantum mechanics would pose nontrivial conceptual problems [14]. We plan to investigate this topic in a future paper.

As in the relativistic case, also the properties of the nonrelativistic Snyder model strongly depend on the sign of the coupling constant λ. If $\lambda > 0$, the momenta are allowed to take any real value, but in the quantum theory a minimal uncertainty in the positions arises. In particular, in the case of a single spatial dimension, the model reduces to the one introduced in [12] in a different context. If $\lambda < 0$, instead, the modulus of the momentum has an upper bound $1/|\lambda|$, but no minimal uncertainty occurs in the quantum theory. However, in contrast with standard quantum mechanics, states with vanishing position uncertainty have finite momentum uncertainty. Moreover, length, area and volume are quantized for positive λ, but not for $\lambda < 0$.

Another interesting application is the study of the harmonic oscillator. Both in classical and quantum mechanics its solution contains corrections of order λE to the standard case. In particular, the frequency of oscillation is no longer independent from the energy.

It may also be interesting to notice the existence of an alternative realization of the Snyder commutation relations for negative λ, that yields a lower bound for the momentum. Although this possibility is pathological under some respects, we briefly discuss it.

2. CLASSICAL MECHANICS OF THE SNYDER MODEL

We first consider the classical implementation of the Snyder model on phase space. In the relativistic case this has been investigated in several papers from various points of view [8,10,7,15].

2.1 The model

Classically, the nonrelativistic Snyder model can be realized by postulating a non-canonical symplectic structure, with fundamental Poisson brackets

$$\{x_i, x_j\} = \lambda\, J_{ij}, \qquad \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij} + \lambda\, p_i p_j, \tag{2.1}$$

where $J_{ij} = x_i p_j - x_j p_i$ are the generators of the group of rotations, and the sign of the coupling constant λ determines the properties of the model.

In analogy with its relativistic counterpart [1], the model can be derived from a 4-dimensional momentum space, constraining the momenta to live on a 3-dimensional hypersurface. This construction is the same as that of de Sitter space, but in momentum space.

For example, the momentum space of the original Snyder model, with $\lambda = \beta^2 > 0$, can be represented as a 3-sphere of radius $1/\beta$ embedded in 4-dimensional Euclidean space of coordinates P_a, with $a = 1, \dots, 4$. The points on the sphere satisfy the equation $P_a^2 = 1/\beta^2$. The construction can be extended to the full phase space [15], by introducing the 4-dimensional coordinates X_a satisfying canonical Poisson brackets with the momenta P_a.

Choosing projective coordinates p_i on the 3-sphere,

$$p_i = \frac{P_i}{\beta P_4} = \frac{P_i}{\sqrt{1 - \beta^2 P_k^2}}, \tag{2.2}$$

where $P_k^2 < 1/\beta^2$, with inverse transformations

$$P_i = \frac{p_i}{\sqrt{1 + \beta^2 p_k^2}}, \qquad \beta P_4 = \frac{1}{\sqrt{1 + \beta^2 p_k^2}}, \tag{2.3}$$

and defining 3-dimensional position coordinates $x_i = \sqrt{1 - \beta^2 P_k^2}\, X_i$, that transform covariantly with respect to the p_i, one obtains the Poisson brackets of the Snyder model, namely

$$\{x_i, x_j\} = \beta^2 J_{ij}, \qquad \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij} + \beta^2 p_i p_j. \tag{2.4}$$

The momentum components p_i so defined range over all real values. Of course, the transformations relating the coordinates X_i, P_i with x_i, p_i are not canonical.

Similarly, the anti-Snyder model, with $\lambda = -\beta^2 < 0$, can be obtained by embedding a 3-dimensional two-sheeted hyperboloid of equation $P_k^2 - P_4^2 = -1/\beta^2$ in 4-dimensional momentum space with Minkowskian signature.

Choosing again projective coordinates

$$p_i = \frac{P_i}{\beta P_4} = \frac{P_i}{\sqrt{1 + \beta^2 P_k^2}}, \tag{2.5}$$

with inverse

$$P_i = \frac{p_i}{\sqrt{1 - \beta^2 p_k^2}}, \qquad \beta P_4 = \frac{1}{\sqrt{1 - \beta^2 p_k^2}}, \tag{2.6}$$

and defining $x_i = \sqrt{1 + \beta^2 P_k^2}\, X_i$, one obtains the Poisson brackets

$$\{x_i, x_j\} = -\beta^2 J_{ij}, \qquad \{p_i, p_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij} - \beta^2 p_i p_j. \tag{2.7}$$

In this case, the momenta are bounded by the relation $p_i^2 < 1/\beta^2$, like in some models of doubly special relativity.

In principle, the Poisson brackets (2.7) may also be derived from a one-sheeted hyperboloid $P_k^2 - P_4^2 = 1/\beta^2$ embedded in in 4-dimensional momentum space with Minkowskian signature. We shall call the resulting model pro-Snyder. In this case, the momentum space has not constant curvature, but this fact has no relevance for our considerations. However, as we shall see, this model suffers some pathologies caused by the fact that in the limit

$\beta \to 0$, the momentum becomes imaginary, so that the kinetic energy has the wrong sign. Therefore we shall not study it in detail, but only give some hints on its properties.

The relations between p_i and P_i are in this case

$$p_i = \frac{P_i}{\sqrt{\beta^2 P_k^2 - 1}}, \tag{2.8}$$

with inverse

$$P_i = \frac{p_i}{\sqrt{\beta^2 p_k^2 - 1}}, \qquad \beta P_4 = \frac{1}{\sqrt{\beta^2 p_k^2 - 1}}, \tag{2.9}$$

and the position coordinates are defined as $x_i = \sqrt{\beta^2 P_k^2 - 1}\ X_i$. Now, both p_k^2 and P_k^2 possess a lower bound, given by $1/\beta^2$. In a relativistic extension of the model, this would lead to a new kind of DSR, with the momenta displaying a lower bound.

Summarizing, we have shown the possibility of defining three different models that obey the Snyder algebra, and differ in the range of definition of the momentum. While in the first case the momenta can take any real value, in the other cases they have an upper or a lower bound, respectively. The latter models can be interpreted in the framework of DSR, where analogous bounds on the allowed values of the momentum occur.

2.2 Symmetries

While the momentum space is by construction invariant under $SO(4)$ or $SO(3,1)$, depending on the sign of λ, from a physical point of view the spatial symmetries of the model, that act on the position coordinates, are more relevant. Let us therefore consider in detail the transformation rules of the full phase space variables.

The phase space coordinates transform as vectors under the action of the generators of rotations $J_{ij} = x_i p_j - x_j p_i = X_i P_j - X_j P_i$,

$$\{J_{ij}, x_k\} = \delta_{ik} x_j - \delta_{ij} x_k, \qquad \{J_{ij}, p_k\} = \delta_{ik} p_j - \delta_{ij} p_k, \tag{2.10}$$

while the Poisson brackets (2.1) transform covariantly. The symmetry under rotations is therefore realized in the usual way.

The Poisson brackets are instead not covariant under ordinary translations. In order to preserve their covariance, the translation symmetry must be realized in a nonlinear and momentum-dependent way, as in DSR [10]. This fact gives rise to some ambiguity in the definition of the generators of translations [7]. The simplest choice is to identify the generators T_i with the momenta p_i, leading, for a translation of infinitesimal parameter a_i, to the relation

$$\delta x_i = a_j \{x_i, p_j\} = a_i + \lambda\, a_j p_j p_i. \tag{2.11}$$

A different choice is possible and is particularly useful in the context of DSR, since it gives rise to deformed dispersion relations. In this case, one identifies the translation generators with the variables P_i, and hence

$$\delta x_i = a_j \{x_i, P_j\} = \frac{a_i}{\sqrt{1 + \lambda p_i^2}}\,. \tag{2.12}$$

For both choices, the momenta p_i are of course unaffected by the transformation.

2.3 Classical motion

The Hamiltonian for a free particle can be defined as the square of the translation generator. For the choice $T_i = p_i$, it has the usual form

$$H = \frac{p_i^2}{2m}. \tag{2.13}$$

If $T_i = P_i$, instead,

$$H = \frac{P_i^2}{2m} = \frac{1}{2m}\ \frac{p_i^2}{1+\lambda p_k^2}. \tag{2.14}$$

Both expressions are invariant under rotations and translations.

The field equations are obtained taking care of the deformed Poisson structure. The Hamiltonian (2.13) yields

$$\dot{x}_i = (1+\lambda p_k^2)\, p_i, \qquad\qquad \dot{p}_i = 0, \tag{2.15}$$

while (2.14) gives

$$\dot{x}_i = \frac{p_i}{1+\lambda p_k^2}, \qquad\qquad \dot{p}_i = 0. \tag{2.16}$$

In both cases, the relation between velocity and momentum is no longer linear. However, the solutions are the classical ones, $p_i = \text{const}$, $x_i = \alpha t + \beta$.

Since the second possibility is more interesting in a relativistic context, in the following we shall limit our considerations to the form (2.13) of the kinetic term of the Hamiltonian.

A nontrivial example of dynamics is that of a one-dimensional harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2 x^2}{2}. \tag{2.17}$$

For unit mass, the Hamilton equations read

$$\dot{x} = (1+\lambda p^2)\, p, \qquad\qquad \dot{p} = -\omega_0^2 (1+\lambda p^2)\, x. \tag{2.18}$$

Let us first consider the Snyder case, $\lambda = \beta^2 > 0$. The equations (2.18) can be solved as follows: the second one can be written

$$\frac{d(\arctan \beta p)}{dt} = -\omega_0^2 \beta x. \tag{2.19}$$

Defining $\bar{p} = \arctan \beta p$, deriving (2.19), and substituting the first Hamilton equation, one obtains

$$\frac{d^2 \bar{p}}{dt^2} = -\omega_0^2 \frac{\sin \bar{p}}{\cos^3 \bar{p}}, \tag{2.20}$$

which admits the first integral

$$\frac{1}{2}\left(\frac{d\bar{p}}{dt}\right)^2 + \frac{\omega_0^2}{2}\ \tan^2 \bar{p} = \text{const} = \omega_0^2 \beta^2 E, \tag{2.21}$$

where the integration constant E has be chosen so that it coincides with the total energy of the oscillator. Clearly, these equations are identical to those of a classical particle moving in the effective potential $V = \omega_0^2 \tan^2 \bar{p}$, depicted in fig. 1.

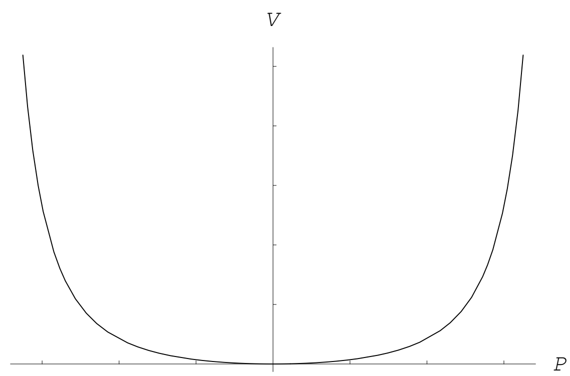

Fig. 1: The effective potential for the Snyder oscillator.

The integration of (2.21) yields

$$\sin\left(\sqrt{1+2\beta^2 E}\;\omega_0 t\right) = \sqrt{\frac{1+2\beta^2 E}{2\beta^2 E}}\;\sin\bar{p}, \tag{2.22}$$

and hence

$$p = \frac{\sqrt{2E}\;\sin\left(\sqrt{1+2\beta^2 E}\;\omega_0 t\right)}{\sqrt{1+2\beta^2 E\cos^2\left(\sqrt{1+2\beta^2 E}\;\omega_0 t\right)}}. \tag{2.23}$$

From (2.18) it is then easy to obtain for x

$$x = \frac{\sqrt{2E(1+2\beta^2 E)}\;\cos\left(\sqrt{1+2\beta^2 E}\;\omega_0 t\right)}{\omega_0\;\sqrt{1+2\beta^2 E\cos^2\left(\sqrt{1+2\beta^2 E}\;\omega_0 t\right)}}. \tag{2.24}$$

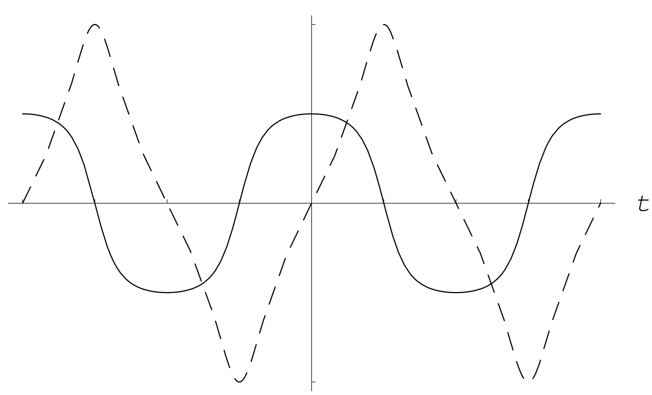

Fig. 2: The solution of the Snyder oscillator for $\beta^2 E$=3. The solid line represents the coordinate x, the dashed line the coordinate p.

The harmonic oscillator has therefore a different solution than in classical mechanics. The solution is still periodic, but the frequency ω presents energy-dependent corrections of order $\beta^2 E$, $\omega = \sqrt{1+2\beta^2 E}\,\omega_0$. Also the amplitude acquires corrections of the same order of magnitude and is no longer sinusoidal (see fig. 2).

In the anti-Snyder case, $\lambda = -\beta^2 < 0$, the solution can be obtained in an analogous way. Writing the second Hamilton equation as

$$\frac{d(\operatorname{arctanh}\beta p)}{dt} = -\omega_0^2 \beta x, \tag{2.25}$$

and defining the variable $\bar{p} = \operatorname{arctanh}\beta p$, one goes through the same steps as before. In particular, one has

$$\frac{d^2\bar{p}}{dt^2} = -\omega_0^2 \frac{\sinh\bar{p}}{\cosh^3\bar{p}}, \tag{2.26}$$

with first integral

$$\frac{1}{2}\left(\frac{d\bar{p}}{dt}\right)^2 + \frac{\omega_0^2}{2}\tanh^2\bar{p} = \text{const} = \omega_0^2\beta^2 E, \tag{2.27}$$

where the integration constant E has be chosen so that it vanishes at the minimum of the potential and coincides with the total energy of the oscillator. From the condition $p^2 < 1/\beta^2$, it follows that $\beta^2 E < \frac{1}{2}$. In this case the equations of motion coincide with those of a particle in the potential $V = \omega_0^2 \tanh^2\bar{p}$, depicted in fig. 3.

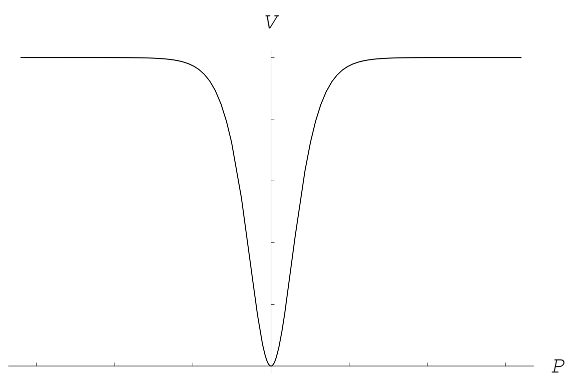

Fig. 3: The effective potential for the anti-Snyder oscillator.

After performing the integration of eq. (2.27), the final result turns out to be the analytic continuation of the solution (2.23)-(2-24) for $\beta^2 \to -\beta^2$, namely

$$p = \frac{\sqrt{2E}\ \sin\left(\sqrt{1-2\beta^2 E}\ \omega_0 t\right)}{\sqrt{1-2\beta^2 E\cos^2\left(\sqrt{1-2\beta^2 E}\ \omega_0 t\right)}}, \tag{2.28}$$

and

$$x = \frac{\sqrt{2E(1-2\beta^2 E)}\ \cos\left(\sqrt{1-2\beta^2 E}\ \omega_0 t\right)}{\omega\ \sqrt{1-2\beta^2 E\cos^2\left(\sqrt{1-2\beta^2 E}\ \omega_0 t\right)}}. \tag{2.29}$$

The properties of the solutions are analogous to those found in the Snyder case (see fig. 4).

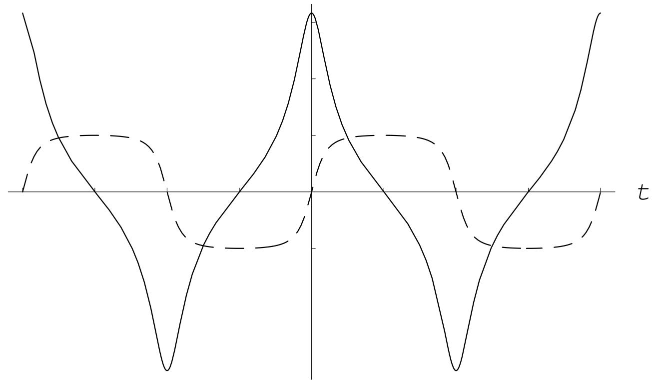

Fig. 4: The solution of the anti-Snyder oscillator for $\beta^2 E = 0.9$. The solid line represents the coordinate x, the dashed line the coordinate p.

Finally, let us briefly consider the pro-Snyder case. Defining $\bar{p} = \operatorname{arccoth} \beta p$, one obtains

$$\frac{1}{2}\left(\frac{d\bar{p}}{dt}\right)^2 + \frac{\omega_0^2}{2}\coth^2\bar{p} = \text{const} = \omega_0^2\beta^2 E, \tag{2.30}$$

with $\beta^2 E > \frac{1}{2}$. In this case the effective potential $V = \omega_0^2 \coth^2 \bar{p}$ has no minimum and therefore the motion is not bounded (see fig. 5). The solution can in fact be written in terms of hyperbolic functions. Intuitively, this behavior can be related to the fact that, as observed in sect. 2.1, in the limit $\beta \to 0$ the kinetic energy of the pro-Snyder model has the wrong sign. Indeed, periodic solutions are available if $\omega_0^2 < 0$.

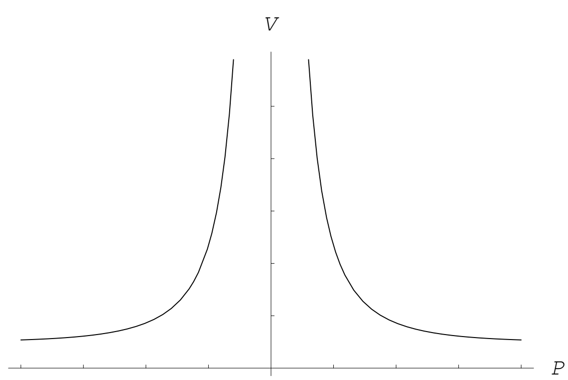

Fig. 5: The effective potential for the pro-Snyder oscillator.

3. QUANTUM MECHANICS OF THE SNYDER MODEL

As it is well known, when passing from classical to quantum mechanics, the Poisson brackets go to commutators, and the deformation of the latter implies a modification of Heisenberg uncertainty relations. Modified uncertainty relations have been studied in several papers [16]. In the following, we shall employ the methods introduced in [12], in order to study the specific deformation induced by the nonrelativistic Snyder model.

3.1 The Snyder model

For Snyder space, the commutation relations between the position operators $\hat{x}_i$ and the momentum operators $\hat{p}_i$ read

$$[\hat{x}_i, \hat{x}_j] = i\hbar\beta^2 \hat{J}_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar(\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j). \tag{3.1}$$

From the results of sect. 2, it is easy to see that they can be realized introducing auxiliary operators $\hat{X}_i$ and $\hat{P}_i$ obeying canonical commutation relations, and performing the nonlinear and nonunitary transformations

$$\hat{x}_i = \sqrt{1 - \beta^2 \hat{P}_k^2}\; X_i, \qquad \hat{p}_i = \frac{\hat{P}_i}{\sqrt{1 - \beta^2 \hat{P}_k^2}}. \tag{3.2}$$

The spectrum of $\hat{P}_i$ must be bounded by $P_k^2 < 1/\beta^2$.

The uncertainty relations following from (3.1) are

$$\Delta x_i \Delta p_j \geq \frac{1}{2} \, | \, \langle [\hat{x}_i, \hat{p}_j] \rangle | = \frac{\hbar}{2} \, [\delta_{ij} + \beta^2 \Delta p_i \Delta p_j + \beta^2 \langle \hat{p}_i \rangle \langle \hat{p}_j \rangle], \tag{3.3}$$

where $\langle \; \rangle$ denotes expectation values.

We first consider the one-dimensional case. In one dimension, the uncertainty relations reduce to

$$\Delta x \Delta p \geq \frac{1}{2} \, | \, \langle [\hat{x}, \hat{p}] \rangle | = \frac{\hbar}{2} [1 + \beta^2 (\Delta p)^2 + \beta^2 \langle \hat{p} \rangle^2]. \tag{3.4}$$

These generalized uncertainty relations have been thoroughly studied in ref. [12], where it was shown that they imply the existence of a minimal position uncertainty, given by

$$\Delta x_M = \hbar\beta \sqrt{1 + \beta^2 \langle \hat{p} \rangle^2}. \tag{3.5}$$

Its minimum value is obtained when $\langle \hat{p} \rangle = 0$, as

$$\Delta x_0 = \hbar\beta, \tag{3.6}$$

in which case $\Delta p = 1/\beta$.

Exploiting (3.2), we can define the position and momentum operators $\hat{p}$ and $\hat{x}$ acting on functions defined on a momentum space parametrized by P, as

$$\hat{p}\,\psi(P) = p\,\psi(P) = \frac{P}{\sqrt{1 - \beta^2 P^2}}\; \psi(P), \qquad \hat{x}\,\psi(P) = i\hbar\sqrt{1 - \beta^2 P^2}\; \frac{\partial \psi(P)}{\partial P}. \tag{3.7}$$

The range of allowed values of P is bounded by $P^2 < 1/\beta^2$, but the spectrum of the momentum operator $\hat{p}$ is unbounded. In [12] a different representation of the same commutation relations was discussed.

In order to define symmetric operators, i.e.

$$(\hat{p}\,\psi, \phi) = (\psi, \hat{p}\,\phi), \qquad (\hat{x}\,\psi, \phi) = (\psi, \hat{x}\,\phi), \tag{3.8}$$

the scalar product must be defined as

$$(\psi, \phi) = \int_{-1/\beta}^{1/\beta} \frac{dP}{\sqrt{1-\beta^2 P^2}} \; \psi^*(P)\, \phi\,(P), \tag{3.9}$$

and the wave functions must be such that $\phi(1/\beta) = \phi(-1/\beta)$. In fact,

$$\begin{aligned} &\int_{-1/\beta}^{1/\beta} \frac{dP}{\sqrt{1-\beta^2 P^2}} \; \psi^*(P)\, i\hbar\sqrt{1-\beta^2 P^2} \; \partial_p \phi\,(P) \\ &= i\hbar\, \psi^* \phi \,\Big|_{-1/\beta}^{1/\beta} + \int_{-1/\beta}^{1/\beta} \frac{dP}{\sqrt{1-\beta^2 P^2}} \; [i\hbar\sqrt{1-\beta^2 P^2} \; \partial_p \psi\,(P)]^* \phi(P). \end{aligned} \tag{3.10}$$

We pass now to study the spectrum of the position operator. The position eigenfunctions ψ_x, with eigenvalue x, are determined by the equation

$$i\hbar\sqrt{1-\beta^2 P^2} \; \frac{\partial \psi_x}{\partial P} = x\, \psi_x, \tag{3.11}$$

whose solution is

$$\psi_x = C \, \exp\left[-\frac{ix}{\hbar\beta} \; \arcsin \beta P\right]. \tag{3.12}$$

The solutions are normalizable, since

$$|\psi_x|^2 = |C|^2 \int_{-1/\beta}^{1/\beta} \frac{dP}{\sqrt{1-\beta^2 P^2}} = \frac{\pi}{\beta} |C|^2. \tag{3.13}$$

and one can set $C = \sqrt{\frac{\beta}{\pi}}$. Moreover, the boundary term in (3.10) cancels only if $x = 2n\hbar\beta$, with integer n, and the spectrum of the position operator is therefore discrete.

As discussed in [12] for a different representation, these states are not physical, since the uncertainty relations (3.4) imply that there are no states with a definite value of the position. Indeed, the expectation values of the momentum $\hat{p}$ and of the kinetic energy $\hat{p}^2/2m$ in the states (3.12) diverge. One can nevertheless calculate the scalar product,

$$(\psi_x, \psi_{x'}) = \frac{\beta}{\pi} \int_{-1/\beta}^{1/\beta} \frac{\exp\left[-\frac{i(x-x')}{\hbar\beta} \; \arcsin \beta P\right]}{\sqrt{1-\beta^2 P^2}} \, dP = \frac{\hbar\beta}{\pi(x-x')} \sin\left[\frac{\pi(x-x')}{2\hbar\beta}\right], \tag{3.14}$$

which shows that the states satisfying the correct boundary conditions are orthogonal.

As proposed in [12], a more relevant basis for the position operator can be obtained by considering states of maximal localization ϕ_x, i.e. states having minimal uncertainty Δx_0 around the position x. These states satisfy the equation

$$\left(\hat{x} \; - \langle\hat{x}\rangle + \frac{\langle[\hat{x},\hat{p}]\rangle}{2(\Delta p)^2} \, (\hat{p} \; - \langle\hat{p}\rangle)\right) \phi_x = 0, \tag{3.15}$$

and are obtained when $\langle\hat{p}\rangle = 0$, $\Delta p = 1/\beta$. Therefore, in our representation, they obey the differential equation

$$\left(i\hbar\sqrt{1-\beta^2P^2}\ \frac{\partial}{\partial P} - x + \frac{i\hbar\beta^2 P}{\sqrt{1-\beta^2P^2}} \right) \phi_x = 0, \tag{3.16}$$

which has solution

$$\phi_x = N\,\sqrt{1-\beta^2P^2}\ \mathrm{e}^{-\frac{ix}{\hbar\beta}\ \arcsin\beta P}, \tag{3.17}$$

with normalization constant $N = \sqrt{\frac{2\beta}{\pi}}$. Also these states must satisfy the condition $x = 2n\hbar\beta$ in order to define a symmetric position operator, but contrary to the functions (3.12), they have finite expectation value of the momentum and the kinetic energy, and are therefore physically meaningful.

The orthogonality properties of this basis can be obtained by calculating the integral

$$\begin{aligned}(\phi_x, \phi_{x'}) &= \frac{2\beta}{\pi} \int_{-1/\beta}^{1/\beta} \sqrt{1-\beta^2P^2}\ \mathrm{e}^{-\frac{i(x-x')}{\hbar\beta}\ \arcsin\beta P}\, dP \\ &= \frac{2\hbar\beta}{\pi(x-x')}\ \sin\frac{\pi(x-x')}{2\hbar\beta}\ \left[1-\left(\frac{\pi(x-x')}{2\hbar\beta}\right)^2\right]^{-1}.\end{aligned} \tag{3.18}$$

Curiously, this result is the same as that obtained in [12] using a different representation of the operators. However, like the states (3.12), also the maximally localized states that satisfy the correct boundary conditions are orthogonal.

3.2 The anti-Snyder model

When $\lambda = -\beta^2 < 0$, the commutation relations read

$$[\hat{x}_i, \hat{x}_j] = -i\hbar\beta^2 \hat{J}_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{x}_i, \hat{p}_j] = i\hbar(\delta_{ij} - \beta^2\hat{p}_i\hat{p}_j). \tag{3.19}$$

and, in analogy with the previous case, can be realized by defining

$$\hat{x}_i = \sqrt{1+\beta^2\hat{P}_k^2}\ \hat{X}_i, \qquad \hat{p}_i = \frac{\hat{P}_i}{\sqrt{1+\beta^2\hat{P}_k^2}}, \tag{3.20}$$

where $\hat{X}_i$ and $\hat{P}_i$ obey canonical commutation relations. From the definition follows that the spectrum of the momentum is bounded from above, $p_k^2 < 1/\beta^2$. In a relativistic context, this would give a realization of the DSR axioms [4]. Actually, several DSR models imply an upper bound for the spectrum of the momentum.

Let us again consider the one-dimensional model. The uncertainty relations are

$$\Delta x \Delta p \geq \frac{1}{2}\,|\,\langle[\hat{x},\hat{p}]\rangle| = \frac{\hbar}{2}\ \left|1-\beta^2(\Delta p)^2 - \beta^2\langle\hat{p}\rangle^2\right|. \tag{3.21}$$

In this case there is no minimal position uncertainty. However, contrary to standard quantum mechanics, the states with vanishing position uncertainty have finite momentum uncertainty, given by $\Delta p = 1/\beta$.

In analogy with the previous case, the position and momentum operators $\hat{p}$ and $\hat{x}$ can be realized on functions defined on a momentum space parametrized by P, as

$$\hat{p}\,\psi(P) = p\,\psi(P) = \frac{P}{\sqrt{1+\beta^2P^2}}\,\psi(P), \qquad \hat{x}\,\psi(P) = i\hbar\sqrt{1+\beta^2P^2}\,\frac{\partial\psi(P)}{\partial P}. \tag{3.22}$$

The scalar product for which the operators (3.22) are symmetric is given by

$$(\psi,\phi) = \int_{-\infty}^{\infty} \frac{dP}{\sqrt{1+\beta^2P^2}}\,\psi^*(P)\,\phi\,(P). \tag{3.23}$$

The position eigenfunctions ψ_x with eigenvalue x are determined by the equation

$$i\hbar\sqrt{1+\beta^2P^2}\,\frac{\partial\psi_x}{\partial P} = x\,\psi_x, \tag{3.24}$$

whose solution is

$$\psi_x = C\,\exp\left[-\frac{ix}{\hbar\beta}\,\mathrm{arcsinh}\,\beta P\right]. \tag{3.25}$$

The properties of these eigenfunctions are analogous to those holding in standard quantum mechanics. They are not normalizable, since

$$|\psi_x|^2 = |C|^2\int_{-\infty}^{\infty}\frac{dP}{\sqrt{1+\beta^2P^2}} \to \infty. \tag{3.26}$$

Nevertheless, setting $C = 1/\sqrt{2\pi\hbar}$, the scalar product of position eigenstates gives

$$(\psi_x,\psi_{x'}) = \delta(x-x'). \tag{3.27}$$

The functions (3.25) are now physical eigenstates, with vanishing Δx, and finite expectation value for p and p^2, but, contrary to ordinary quantum mechanics, they exhibit finite momentum uncertainty, $\Delta p = 1/\beta$, as can be readily checked.

3.3 The pro-Snyder model

As we have seen for the classical theory, the commutation relations of the anti-Snyder model can be realized also in an alternative way, by defining

$$\hat{x}_i = \sqrt{\beta^2\hat{P}_k^2-1}\;\hat{X}_i, \qquad \hat{p}_i = \frac{\hat{P}_i}{\sqrt{\beta^2\hat{P}_k^2-1}} \tag{3.28}$$

where $\hat{X}_i$ and $\hat{P}_i$ obey canonical commutation relations and the spectrum of P_i satisfies the bound $P_k^2 > 1/\beta^2$. Also the momentum p_i has a lower bound, $p_k^2 > 1/\beta^2$. This setting is similar to the previous one, and its relativistic generalization may be considered as a different kind of DSR, with lower bounds for the momenta.

Since the commutation relations are the same as in the anti-Snyder case, also the uncertainty relations enjoy the same properties. In one dimension, the position and momentum operators $\hat{p}$ and $\hat{x}$ can be realized on functions defined on a momentum space parametrized by P, as

$$\hat{p}\,\psi(P) = p\,\psi(P) = \frac{P}{\sqrt{\beta^2 P^2 - 1}}\,\psi(P), \qquad \hat{x}\,\psi(P) = i\hbar\sqrt{\beta^2 P^2 - 1}\,\frac{\partial\psi(P)}{\partial P}, \tag{3.29}$$

and a suitable scalar product is given by

$$(\psi, \phi) = \int_{|P|\geq 1/\beta} \frac{dP}{\sqrt{\beta^2 P^2 - 1}}\,\psi^*(P)\,\phi\,(P), \tag{3.30}$$

The position eigenfunctions ψ_x with eigenvalue x are determined by the equation

$$i\hbar\sqrt{\beta^2 P^2 - 1}\,\frac{\partial\psi_x}{\partial P} = x\,\psi_x$$

whose solution is

$$\psi_x = C\,\exp\left[-\frac{ix}{\hbar\beta}\,\mathrm{arccosh}\,\beta P\right]. \tag{3.31}$$

The eigenfunctions are not normalizable, since

$$|\psi_x|^2 = |C|^2 \int_{|P|\geq 1/\beta} \frac{dP}{\sqrt{\beta^2 P^2 - 1}} \to \infty, \tag{3.32}$$

but setting $C = 1/\sqrt{2\pi\hbar}$, the scalar product of position eigenstates gives

$$(\psi_x, \psi_{x'}) = \delta(x - x'). \tag{3.33}$$

However, contrary to the previous case, the expectation values of $\hat{p}$ and $\hat{p}^2$ diverge for the states (3.31), due to the singularity at $P^2 = 1/\beta^2$. This model appears therefore to be unviable.

3.4 Quantum symmetries

The invariance of the classical model under rotations and translations can be extended to the quantum case.

The rotations are generated by

$$\hat{J}_{ij} = \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i = i\hbar\left(P_j\frac{\partial}{\partial P_i} - P_i\frac{\partial}{\partial P_j}\right). \tag{3.34}$$

and act in the standard way. In particular, the spectrum of $\hat{J}_{ij}$ is the same as in ordinary quantum mechanics.

As in the classical Snyder model, also in the quantum case the definition of translations is ambiguous, since their action is nonlinear. In particular, one may take as generators of translations the momentum operators $\hat{p}_i$, whose action on the positions operators is deformed, due to the commutation relations (3.1). Analogously, in some problems it may be more convenient to identify the translation generators with the $\hat{P}_i$.

Anyway, the commutation relations (3.1) transform covariantly under the symmetries defined above.

3.5 Position eigenstates in three dimensions

The analysis of the position operators of the previous sections easily generalizes to three dimensions. It must be taken into account, however, that the position operators $\hat{x}_i$ do not commute. For a free particle, the most relevant observable is the rotation-invariant radial coordinate $\hat{r} = \sqrt{\hat{x}_i^2}$, and therefore we adopt radial coordinates.

As before, we work in a momentum representation. A basis of operators for one-particle states is given by the radial momentum $\hat{p}_r = \sqrt{\hat{p}_i^2}$ and the angular momentum $\hat{L}_i = \epsilon_{ijk}\hat{J}_{jk}$.

Let us consider for example the Snyder model. Defining

$$\hat{p}_r\psi = \frac{P_r}{\sqrt{1-\beta^2P_r^2}}\ \psi, \qquad \hat{r}\,\psi = i\hbar\sqrt{1-\beta^2P_r^2}\left(\frac{\partial}{\partial P_r}+\frac{1}{P_r}\right)\psi, \tag{3.35}$$

one has

$$[\hat{r}, \hat{p}_r] = i\hbar(1+\beta^2p_r^2). \tag{3.36}$$

The uncertainty relations for the radial coordinates are therefore identical to those of the one-dimensional particle and enjoy the same properties.

Since the angular momentum action is the standard one, the wave function for a free particle can be expanded in spherical harmonics,

$$\psi_{rlm}(P_r, P_\theta, P_\phi) = \psi_r(P_r)\,Y_{lm}(P_\theta, P_\phi), \tag{3.37}$$

and we only need to investigate the radial functions. Their scalar product can be defined as

$$(\psi_r, \phi_r) = \int_0^{1/\beta} \frac{P_r^2\, dP_r}{\sqrt{1-\beta^2P_r^2}}\ \psi_r^*(P_r)\,\phi_r\,(P_r). \tag{3.38}$$

Since $\hat{p}$ acts by multiplication, its spectrum is trivial. The spectrum of the radial position operator is instead obtained from the equation

$$i\hbar\sqrt{1-\beta^2P_r^2}\left(\frac{\partial}{\partial P_r}+\frac{1}{P_r}\right)\psi = r\psi, \tag{3.39}$$

whose solution is

$$\psi_x(P_r) = C\ \frac{\mathrm{e}^{-\frac{ir}{\hbar\beta}\ \arcsin\beta P_r}}{P_r}, \tag{3.40}$$

with C an integration constant. The $1/P_r$ factor in the eigenfunctions cancels the P_r^2 in the scalar product (3.38), so that the orthogonality properties of the eigenfunctions are the same as in one dimension, except that the integrals are restricted to positive values of P_r. We shall therefore not repeat the discussion of sect. 3.1. We just recall that these are not physical states, since the expectation value of the energy diverges.

One can however introduce maximally localized states, that satisfy the equation

$$\left[i\hbar\sqrt{1-\beta^2P_r^2}\left(\frac{\partial}{\partial P_r}+\frac{1}{P_r}\right)-r+i\hbar\beta^2\frac{P_r}{\sqrt{1-\beta^2P_r^2}}\right]\phi_r=0, \tag{3.41}$$

which has solutions

$$\phi_x=N\ \frac{\sqrt{1-\beta^2P_r^2}}{P_r}\ \mathrm{e}^{-\frac{ir}{\hbar\beta}\ \arcsin\beta P_r}. \tag{3.42}$$

Also the properties of the maximally localized eigenfunctions reproduce those of their one-dimensional analogues.

The basis adopted in this section also permits to immediately find the states that minimize the uncertainty relations between the position coordinates along different directions. In fact, from (3.1) it follows that

$$\Delta x_i\Delta x_j\geq\frac{\hbar\beta^2}{2}\ |\langle J_{ij}\rangle|, \tag{3.43}$$

and the states that minimize these uncertainty relations are those with vanishing angular momentum.

The previous discussion can be easily extended to the anti-Snyder model.

3.6 The Schrödinger equation

The Schrödinger equation for free particles in momentum space is algebraic and in our case gives rise to the dispersion relation, that relates the energy to the momentum. It reads

$$\frac{\hat p_k^2}{2m}\ \psi=\frac{p_k^2}{2m}\ \psi=E\psi, \tag{3.44}$$

yielding the usual relation between momentum and energy. If one adopts instead the definition $H=\hat P_k^2/2m$, the equation becomes

$$\frac{\hat P_k^2}{2m}\ \psi=\frac{1}{2m}\ \frac{p_k^2}{1+\lambda p_k^2}\ \psi=E\psi, \tag{3.45}$$

leading to a deformed dispersion relation. In the following, we shall not consider this possibility.

A less trivial problem is given by the one-dimensional harmonic oscillator, with Hamiltonian

$$H=\frac{\hat p^2}{2m}+\frac{m\omega_0^2\hat x^2}{2}. \tag{3.46}$$

For unit mass, the Schrödinger equation for the Snyder oscillator is, in the representation (3.7),

$$\frac{d^2\psi}{dP^2} - \frac{\beta^2 P}{1-\beta^2P^2}\frac{d\psi}{dP} - \frac{1}{\hbar^2\omega_0^2}\left[\frac{P^2}{(1-\beta^2P^2)^2} - \frac{2E}{1-\beta^2P^2}\right]\psi = 0, \tag{3.47}$$

with $P^2 < 1/\beta^2$. In terms of a variable $\bar{P} = \arcsin\beta P$, this becomes the standard Schrödinger equation for a potential

$$V = \frac{1}{\omega_0^2}\ \tan^2\bar{P}, \tag{3.48}$$

which coincides with classical potential of sect. 2.3.

In order to find the explicit solution of eq. (3.47), it is however more convenient to define the variable $z = (1+\beta P)/2$, in terms of which the equation can be written in the standard hypergeometric form

$$\frac{d^2\psi}{dz^2} + \frac{z-\frac{1}{2}}{z(z-1)}\frac{d\psi}{dz} - \left[\frac{\mu(z-\frac{1}{2})^2}{z^2(z-1)^2} + \frac{\epsilon}{z(z-1)}\right]\psi = 0, \tag{3.49}$$

with $\mu = 1/\hbar^2\omega_0^2\beta^4$, $\epsilon = 2E/\hbar^2\omega_0^2\beta^2$. The solution reads

$$\psi = \text{const} \times (1-\beta^2P^2)^{(1+\sqrt{1+4\mu})/4}\ F\left(a,b,c;\frac{1+\beta P}{2}\right), \tag{3.50}$$

with

$$a = \frac{1}{2}(1+\sqrt{1+4\mu}) - \sqrt{\mu+\epsilon}, \quad b = \frac{1}{2}(1+\sqrt{1+4\mu}) + \sqrt{\mu+\epsilon}, \quad c = 1 + \frac{1}{2}\sqrt{1+4\mu}\,.$$

We require that ψ vanish at $P = \pm 1/\beta$, i.e. at $z = 0, 1$. This occurs when $a = -n$ or $b = -n$. In both cases,

$$\epsilon = (n+\frac{1}{2})(1+\sqrt{1+4\mu}) + n^2, \tag{3.51}$$

namely,

$$\begin{aligned} E &= \hbar\omega_0\left[\left(n+\frac{1}{2}\right)\left(\sqrt{1+\frac{\hbar^2\omega_0^2\beta^4}{4}} + \frac{\hbar\omega_0\beta^2}{2}\right) + \frac{\hbar\omega_0\beta^2}{2}\,n^2\right] \\ &\approx \hbar\omega_0\left[n+\frac{1}{2}+\frac{\hbar\omega_0\beta^2}{2}\left(n^2+n+\frac{1}{2}\right)\right]. \end{aligned} \tag{3.52}$$

Hence corrections of order $\hbar\omega_0\beta^2$ occur in the spectrum of the harmonic oscillator, and the relation between the ground state energy and ω_0 is no longer linear. The expression (3.52) for the energy is similar to the one obtained in [12] for a different representation of the commutation relations, but there are some differences in the numerical coefficients.

The same calculation can be performed for the anti-Snyder oscillator. The Schrödinger equation reads now

$$\frac{d^2\psi}{dP^2}+\frac{\beta^2P}{1+\beta^2P^2}\ \frac{d\psi}{dP}-\frac{1}{\hbar^2\omega_0^2}\left[\frac{P^2}{(1+\beta^2P^2)^2}+\frac{2E}{1+\beta^2P^2}\right]\psi=0. \tag{3.53}$$

Again one may define a new variable $\bar{P}=\operatorname{arcsinh}\beta P$, in terms of which (3.53) becomes the standard Schrödinger equation for a potential identical to that obtained for the classical motion,

$$V=\frac{1}{\omega_0^2}\ \tanh^2\bar{P}, \tag{3.54}$$

and therefore bound states are possible for $0\le E<\beta^2/2$ (see fig. 3). The latter inequality is also a consequence of the bound on the momentum $p^2<1/\beta^2$.

Defining $z=(1+i\beta P)/2$, eq. (3.53) can be written in the form of a hypergeometric differential equation,

$$\frac{d^2\psi}{dz^2}+\frac{z-\frac{1}{2}}{z(z-1)}\ \frac{d\psi}{dz}-\left[\frac{\mu(z-\frac{1}{2})^2}{z^2(z-1)^2}-\frac{\epsilon}{z(z-1)}\right]\psi=0, \tag{3.55}$$

with $\mu=1/\hbar^2\omega_0^2\beta^4$, $\epsilon=2E/\hbar^2\omega_0^2\beta^2$. The solution is given by

$$\psi=\text{const}\times(1+\beta^2P^2)^{(1-\sqrt{1+4\mu})/4}\ F\left(a,b,c;\frac{1+i\beta P}{2}\right), \tag{3.56}$$

with

$$a=\frac{1}{2}(1-\sqrt{1+4\mu})-\sqrt{\mu-\epsilon},\quad b=\frac{1}{2}(1-\sqrt{1+4\mu})+\sqrt{\mu-\epsilon},\quad c=1-\frac{1}{2}\sqrt{1+4\mu}\,.$$

The function ψ is regular at infinity if $a=-n$ or $b=-n$. In both cases,

$$\epsilon=\left(n+\frac{1}{2}\right)\left(\sqrt{1+4\mu}-1\right)-n^2, \tag{3.57}$$

i. e.

$$\begin{aligned} E&=\hbar\omega_0\left[(n+\frac{1}{2})\left(\sqrt{1+\frac{\hbar^2\omega_0^2\beta^4}{4}}-\frac{\hbar\omega_0\beta^2}{2}\right)-\frac{\hbar\omega_0\beta^2}{2}\,n^2\right]\\ &\approx\hbar\omega_0\left[n+\frac{1}{2}-\frac{\hbar\omega_0\beta^2}{2}\left(n^2+n+\frac{1}{2}\right)\right]. \end{aligned} \tag{3.58}$$

As it could have been guessed, the energy spectrum is simply the analytic continuation of the Snyder one for $\beta\to i\beta$, with analogous properties. However, an important difference arises. In the present case, the energy (3.58) becomes negative for large n. In order to

preserve the bound $E \geq 0$, one must impose that $n \leq \sqrt{\mu + \frac{1}{4}} + \sqrt{\mu} - \frac{1}{2}$, and hence only a finite number of energy levels are present.

For the pro-Snyder model, the Schrödinger equation takes the form

$$\frac{d^2\psi}{dP^2} + \frac{\beta^2 P}{\beta^2 P^2 - 1}\ \frac{d\psi}{dP} - \frac{1}{\hbar^2\omega_0^2}\left[\frac{P^2}{(\beta^2 P^2 - 1)^2} - \frac{2E}{\beta^2 P^2 - 1}\right]\psi = 0, \tag{3.59}$$

with $P^2 > 1/\beta^2$. In terms of the variable $\bar{P} = \operatorname{arccosh} \beta P$, this becomes the standard Schrödinger equation for a potential

$$V = \frac{1}{\omega_0^2}\ \coth^2 \bar{P}, \tag{3.60}$$

as in the classical case. It is evident that this potential does not admit bound states (see fig. 5). An explicit solution of (3.59) can still be obtained in terms of hypergeometric functions, but we shall not report it here.

3.7 Quantization of area

Another interesting implication of the Snyder model is that it gives rise to the quantization of areas, as was first shown in ref. [13].

This fact can be deduced by noting that every pair of spatial coordinates and their commutator satisfy an $so(3)$ algebra. For example,

$$[\hat{L}_1, \hat{L}_2] = i\hat{L}_3, \qquad [\hat{L}_3, \hat{L}_1] = i\hat{L}_2, \qquad [\hat{L}_2, \hat{L}_3] = i\hat{L}_1, \tag{3.61}$$

where $\hat{L}_1 = \hat{x}_1/\beta$, $\hat{L}_2 = \hat{x}_2/\beta$ and $\hat{L}_3 = \hat{J}_{12}$.

For a disc in the $x_1 x_2$ plane, the area operator is defined as

$$\hat{A} = \pi(\hat{x}_1^2 + \hat{x}_2^2) = \pi\beta^2(\hat{L}^2 - \hat{L}_3^2), \tag{3.62}$$

where $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2$ is the Casimir operator of $so(3)$. Defining as usual

$$\hat{L}^2|\,lm\rangle = l(l+1)|\,lm\rangle, \qquad \hat{L}_3|\,lm\rangle = m|\,lm\rangle, \tag{3.63}$$

for $l = 0, 1, \ldots$, $|m| \leq l$, the spectrum of $\hat{A}$ follows immediately[2],

$$\hat{A}|\,lm\rangle = \pi\beta^2[l(l+1) - m^2]|\,lm\rangle, \tag{3.64}$$

and is therefore discrete.

In the anti-Snyder case, instead, the commutation relations become

$$[\hat{L}_1, \hat{L}_2] = -i\hat{L}_3, \qquad [\hat{L}_3, \hat{L}_1] = i\hat{L}_2, \qquad [\hat{L}_2, \hat{L}_3] = i\hat{L}_1, \tag{3.65}$$

[2] We do not consider spinorial representations.

and the operators span an $so(2,1)$ algebra, which is not compact and hence admits a continuous spectrum.

In fact, the area operator is now

$$\hat{A} = \pi(\hat{x}_1^2 + \hat{x}_2^2) = \pi\beta^2(\hat{L}^2 + \hat{L}_3^2), \tag{3.66}$$

where $\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 - \hat{L}_3^2$ is the Casimir operator of $so(2,1)$. The algebra $so(2,1)$ admits both continuous and discrete spectra [17]. The continuous spectrum is given by

$$\hat{L}^2|\, lm\rangle = l^2|\, lm\rangle, \qquad \hat{L}_3|\, lm\rangle = m|\, lm\rangle, \tag{3.67}$$

with $l > 0$, $m = 0, \pm 1, \pm 2, \ldots$, and hence

$$\hat{A}|\, lm\rangle = \pi\beta^2[l^2 + m^2]|\, lm\rangle. \tag{3.68}$$

The discrete series is given instead by

$$\hat{L}^2|\, lm\rangle = l(1-l)|\, lm\rangle, \qquad \hat{L}_3|\, lm\rangle = m|\, lm\rangle, \tag{3.69}$$

for $l = 1, 2, \ldots$, $|m| \geq l$. In this case,

$$\hat{A}|\, lm\rangle = \pi\beta^2[l(1-l) + m^2]|\, lm\rangle. \tag{3.70}$$

These results can be easily extended to the volume operator [13]. We conclude that, while in the Snyder model positions, areas and volumes are quantized, this does not necessarily occur in the anti-Snyder case.

4. CONCLUSIONS

We have studied the classical and the quantum mechanics of free particles in the nonrelativistic version of the Snyder model. The calculations are based on the existence of a nonlinear transformation relating the Snyder phase space coordinates to canonical coordinates.

As in the relativistic case, that will be treated in detail elsewhere, the results strongly depend on the sign of the coupling constant in the defining Poisson brackets (or commutation relations in the quantum case). In particular, it turns out that the spectrum of length, area and volume operators in the quantum model is not necessarily discrete, in spite of the noncommutativity of the geometry. Furthermore, the range of definition of the momenta depends on the sign of the coupling constant.

Also interesting is the behavior of the one-dimensional harmonic oscillator: both in the classical and in the quantum cases, its frequency is energy dependent. Moreover, if the coupling constant is negative, the quantum spectrum has a finite number of eigenvalues.

Of course, the extension of the present investigations to the relativistic theory is of special importance. This gives rise to interesting conceptual problems [14], in particular in the quantum mechanical case, where a field theory should be defined. The dependence of

the oscillator frequency on the energy may have important implications for the quantum fields.

The present analysis may also be extended to the case of Yang's model [18], where the background space is no longer flat, but has constant curvature, and in particular to its nonlinear realization, called triply special relativity [19,15]. In this case, however, the discovery of a transformation relating the noncanonical phase space variables to canonical ones appears to be more problematic.

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